MATHEMATICS OF COMPUTATION
VOLUME 47, NUMBER 175
JULY 1986, PAGES S1-S8

Supplement to Discontinuous Polynomial Approximations in the Theory of One-Step, Hybrid and Multistep Methods for Nonlinear Ordinary Differential Equations

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APPENDIX 1. POLYNOMIAL SPACES.

Consider the space $P^{K-L}(0,1;E)\times E^L$ endowed with the norm

$$\|\tilde{w}\| = \|w\|_0 + \sum_{\ell=1}^{L} |W_\ell|$$

where $\tilde{w} = (w,W_1,\ldots,W_L)$ and define the L^2-projector $\mathcal{P}_L:L^2(0,1;E) \to P^{K-L}(0,1;E)$ (if L=K+1, set $P^{K-L}(0,1;E) = \{0\}$). Let $\{\tau_\ell\}_{\ell=1}^{L}$ be a set of L distinct points outside the open interval (0,1).

Lemma A1.1. The map $u \to Ju:P^K(0,1;E) \to P^{K-L}(0,1;E)\times E^L$ defined by $Ju = (\mathcal{P}_L u,u(\tau_1),\ldots,u(\tau_L))$ is an isomorphism.

Proof. We only have to show that J is injective. If Ju = 0, u has at least L real roots $(\tau_1,\ldots,\tau_L)$. Thus it can be decomposed as follows:

$$u = \Omega\tilde{u}, \quad \Omega(\tau) = \prod_{\ell=1}^{L} (\tau-\tau_\ell),$$

where $\tilde{u}$ is some polynomial in $P^{K-L}(0,1;E)$ to be determined. But u belongs to $[P^{K-L}(0,1;E)]^\perp$ since $\mathcal{P}_L u = 0$. As a result

$$\int_0^1 \Omega|\tilde{u}|^2 = \int_0^1 u\cdot\tilde{u} = 0$$

and $\tilde{u}=0$ since Ω does not change sign in [0,1]. A fortiori u=0 and J is injective. □

From the above result we know that there exist two strictly positive constants such that

$$\beta_1(\tau_1,\ldots,\tau_L)\|u\|_0 \le \|Ju\| \le \beta_2(\tau_1,\ldots,\tau_L)\|u\|_0.$$

We now show that the two constants β_1 and β_2 can be made independent of the set $\{\tau_\ell\}_{\ell=1}^{L}$.

The hypothesis of regularity of the partition and hypothesis (3.5) suggest to consider the following compact set

$$T_\gamma^c = \left\{(\tau_1,\ldots,\tau_L) \in R^L: \left| \begin{array}{l} (1)\ \tau_\ell \text{ is in } [\gamma,1] \text{ for all } \ell=1,\ldots,L \\ (2)\ \tau_\ell-\tau_{\ell-1} \ge c \quad \text{for all } \ell=2,\ldots,L \end{array} \right. \right\}$$

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where γ is a real constant and c is the constant in (2.4) (we can choose $\gamma = -\frac{M}{c}$ where M is the integer of (3.5)).

Lemma A1.2. There exist two strictly positive constants β_1 and β_2 such that

(1) $\beta_1 \|u\|_0 \le \|Ju\| \le \beta_2 \|u\|_0$

for every application J defined on a set of points in T^c_γ.

Proof. We only have to show that

$$0 < \beta_1 = \inf_{(\tau_1,\ldots,\tau_L)\in T^c_\gamma} \beta_1(\tau_1,\ldots,\tau_L)$$

where $\beta_1(\tau_1,\ldots,\tau_L) = \inf_{\|u\|_0=1} \|Ju\|$.

By a compactness argument, there exists a sequence $\{(\tau_1^i,\ldots,\tau_L^i)\}_{i=1}^\infty$ in T^c_γ such that

$$\beta_1 = \lim_{i\to\infty} \beta_1(\tau_1^i,\ldots,\tau_L^i)$$

and a point $(\tau_1,\ldots,\tau_L)$

$$(\tau_1,\ldots,\tau_L) = \lim_{i\to\infty} (\tau_1^i,\ldots,\tau_L^i)$$

where $(\tau_1,\ldots,\tau_L)$ is in T^c_γ. There exists also a sequence $\{u^i\}_{i=1}^\infty$ in $P^K(0,1;E)$ such that

$$\|u^i\|_0 = 1, \quad \|J^i u^i\| = \beta_1(\tau_1^i,\ldots,\tau_L^i)$$

and

$$u = \lim_{i\to\infty} u^i ,$$

where u is in $P^K(0,1;E)$. Then

$$\mathcal{P}_L u = \lim_{i\to\infty} \mathcal{P}_L u^i$$

and $\{u^i\}_{i=1}^\infty$ converge uniformly to u over $[\gamma,1]$. Thus by uniform continuity and uniform convergence we show that

$$u(\tau_\ell) = \lim_{i\to\infty} u^i(\tau_\ell^i)$$

for $\ell=1,\ldots,L$. Then we have

$$\|Ju\| = \lim_{i\to\infty} \|J^i u^i\| = \beta_1$$

and this shows that $\beta_1(\tau_1,\ldots,\tau_L) = \beta_1$ and that β_1 is strictly positive, since $u \neq 0$.

We obtain β_2 by a similar argument. □

APPENDIX 2. RECURRENT INEQUALITIES.

In the error analysis section we use Lemma A2.2 proved below.

Lemma A2.1. If the sequence $\{u_n\}_{n=0}^\infty$ of positive reals satisfies the recurrent inequality

(1) $u_n \le a_n u_{n-1} + b_n$

for $n=1,2,\ldots$ where $a_n, b_n \ge 0$, then

(2) $u_n \le u_0 \prod_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \prod_{j=i+1}^{n} a_j$

for all n (where $\Pi_{i=n+1}^n a_i = 1$ and $\Sigma_{i=n+1}^n a_i = 0$).

Proof. We obtain (2) by direct substitution of (1) into itself. □

Lemma A2.2. If the sequence $\{u_n\}_{n=0}^\infty$ of positive reals satisfies the recurrent inequality

(3) $u_n \le u_{n-1} + \sum_{m=1}^{M} \alpha_{nm} u_{n-m} + b_n$

where $\alpha_{nm} \ge 0$ for $m=1,\ldots,M$ and $b_n \ge 0$, for $n=M,M+1,\ldots,$ then

(4) $u_n \le \delta \prod_{i=M}^{n} (1+A_i) + \sum_{i=M}^{n} b_i \prod_{j=i+1}^{n} (1+A_j)$

for all n where $A_i = \Sigma_{m=1}^M \alpha_{im}$ and δ is any real number satisfying $\delta \ge u_r$ for $r=0,\ldots,M-1$.

<u>Proof</u>. Let $\{y_n\}_{n=0}^{\infty}$ be the solution of the difference equation

$$y_n = (1+A_n)y_{n-1} + b_n$$

for $n=M,M+1,\dots,$ with $y_r = \delta \ge u_r$ for $r=0,\dots,M-1$. If $y_n \ge u_n$ for $n=0,\dots,N-1$, we have from (3):

$$u_N \le y_{N-1} + \sum_{m=1}^{M} \alpha_{Nm} y_{N-m} + b_N$$

and since $y_n \ge y_{n-1}$ we obtain

$$u_N \le (1+A_N)y_{N-1} + b_N = y_N .$$

This shows that $u_n \le y_n$ for all n and the result follows from Lemma A2.1 applied to $\{y_n\}_{n=0}^{\infty}$. □

APPENDIX 3. QUADRATURE FORMULA.

Consider L distinct points $\tau_1,\dots,\tau_L$ given outside of the open interval (0,1). We are looking for K+1-L distinct points and K+1 weights so that the quadrature formula

$$(1)\qquad \int_0^1 \psi(\tau)d\tau \simeq \sum_{k=1}^{K+1} a_k\psi(\tau_k)$$

be exact for polynomials of degree at most 2K+1-L.

<u>Definition A3.1</u>. A K+1 -point quadrature formula (1) is called <u>interpolatory</u> if it integrates exactly the polynomials of degree at most K. □

Introduce the two polynomials

$$\Omega(\tau) = \prod_{\ell=1}^{L} (\tau-\tau_\ell), \qquad \omega(\tau) = \prod_{k=L+1}^{K+1} (\tau-\tau_k).$$

We obtain the following characterization.

<u>Theorem A3.2</u>. In order that formula (1) be exact for all polynomials of degree $\le$ 2K+1-L it is necessary and sufficient that (i) it be interpolatory, and that (ii) the polynomial ω be orthogonal on [0,1] with respect to Ω to every polynomial of degree $\le$ K-L, in other words

$$\int_0^1 \Omega(\tau)\ \omega(\tau)p(\tau)d\tau = 0$$

for every polynomial p of degree $\le$ K-L.

<u>Proof</u>. See V.I.KRYLOV [24, Chapter 9, Theorem 1]. □

To achieve that goal, we shall prove that there exists a polynomial ω of degree K+1-L such that $\Omega\omega$ belongs to $P^{K+1}(0,1;R) \cap [P^{K-L}(0,1;R)]^{\perp}$ and that ω has K+1-L distinct roots in (0,1).

We must be able to write

$$(2)\qquad \Omega\omega = c_0\pi_{K+1} + \dots + c_L\pi_{K+1-L}$$

where π_ℓ is the Legendre polynomial of degree ℓ, translated and scaled to [0,1], with normalization $\pi_\ell(1) = 1$. Moreover the coefficients of (2) are a solution of the linear homogeneous system of L equations with L+1 unknowns

$$(3)\qquad \Omega\omega(\tau_\ell) = 0 \qquad \ell = 1,\dots,L.$$

Since we are looking for a non-zero polynomial ω, we first show that every non trivial solution of (3) must have $c_0 \ne 0$. If not,

$$\omega = \frac{c_1\pi_K + \dots + c_L\pi_{K+1-L}}{\Omega}$$

is in $P^{K-L}(0,1;R)$ and

$$\int_0^1 \Omega\omega^2 = 0,$$

which is impossible since Ω does not change sign on [0,1]. We also have

$$\begin{bmatrix} c_1 \\ \vdots \\ c_L \end{bmatrix} = -c_0 \begin{bmatrix} \pi_K(\tau_1) & \dots & \pi_{K+1-L}(\tau_1) \\ \vdots & & \vdots \\ \pi_K(\tau_L) & \dots & \pi_{K+1-L}(\tau_L) \end{bmatrix}^{-1} \begin{bmatrix} \pi_{K+1}(\tau_1) \\ \vdots \\ \pi_{K+1}(\tau_L) \end{bmatrix}$$

and we can choose c_0 such that the leading coefficient in ω be equal to one:

$$\omega(\tau) = \tau^{K+1-L} + d_1\tau^{K-L} + \ldots + d_{K+1-L}.$$

The next result locates the roots of ω .

Theorem A3.3. If the polynomial ω of degree K+1-L is orthogonal on the segment [0,1] to all polynomials of degree at most K-L with respect to Ω , then all the roots of ω are real, distinct and lie in (0,1).

Proof. See V.I.KRYLOV [24], chapter 2, Theorem 1. □

To determine the weights, choose

$$\phi_\ell(\tau) = \frac{\omega^2(\tau)\Omega(\tau)}{(\tau-\tau_\ell)\omega^2(\tau_\ell)\Omega'(\tau_\ell)}, \qquad \ell=1,\ldots,L,$$

which are in $P^{2K+1-L}(0,1;R)$. Then

$$a_\ell = \int_0^1 \phi_\ell(\tau)d\tau, \qquad \ell=1,\ldots,L,$$

and a_ℓ are not zero since ϕ_ℓ does not change sign in (0,1). In the same way, define

$$\theta_k(\tau) = \frac{\omega^2(\tau)\Omega(\tau)}{(\tau-\tau_k)^2(\omega'(\tau_k))^2\Omega(\tau_k)}, \qquad k=L+1,\ldots,K+1,$$

and

$$a_k = \int_0^1 \theta_k(\tau)d\tau, \qquad k=L+1,\ldots,K+1.$$

These weights $\{a_k\}_{k=L+1}^{K+1}$ are strictly positive since the θ_k's are positive on (0,1).

APPENDIX 4. COMPLETELY DISCONTINUOUS METHODS (L=0).

This family of examples corresponds to the numerical scheme (3.29) of Corollary 3.8 when L=0. The L^2 and nodal-convergence rates are respectively proportional to h^{K+1} and h^{2K+2}.

Here the τ_i's are the absissae of the (K+1)-point Gauss-Legendre quadrature formula. This formula integrates exactly all polynomials in $P^{2K+1}(0,1)$. The polynomials $\theta_1,\theta_2,\ldots,\theta_{K+1}$ in $P^K(0,1)$ are the Lagrange interpolating polynomials associated with those points:

(1) $$\theta_i(\tau) = \prod_{\substack{j=1 \\ j\neq i}}^{K+1} \frac{\tau-\tau_j}{\tau_i-\tau_j}, \qquad i=1,2,\ldots,K+1,$$

(2) $$\left\{\begin{array}{l} \psi_i(\tau) = \int_\tau^1 \theta_i(\sigma)d\sigma \\ t_{ni} = t_{n-1}+\tau_i h_n \end{array}\right\}, \quad i=1,2,\ldots,K+1.$$

The numerical scheme is given by the following formulae:

(3a) $$u_{nk} = U_{n-1} + h_n \sum_{\ell=1}^{K+1} \frac{a_\ell}{a_k}\psi_k(\tau_\ell)f(u_{n\ell},t_{n\ell}), \qquad k=1,2,\ldots,K+1,$$

(3b) $$U_n = U_{n-1} + h_n \sum_{\ell=1}^{K+1} a_\ell f(u_{n\ell},t_{n\ell}).$$

Case K=0. $\tau_1 = \frac{1}{2}$, $a_1 = 1$.

(4) $$\left\{\begin{array}{ll} u_{n1} = U_{n-1} + h_n\frac{1}{2}f(u_{n1}, t_{n1}), & t_{n1} = t_{n-1} + \frac{1}{2}h_n = \frac{t_n+t_{n-1}}{2}, \\ U_n = U_{n-1} + h_n f(u_{n1},t_{n1}). & \end{array}\right.$$

By eliminating u_{n1} we obtain the Crank-Nicolson process of order 2:

(5) $$U_n = U_{n-1} + h_n f\left(\frac{U_n+U_{n-1}}{2}, \frac{t_n+t_{n-1}}{2}\right).$$

Case K=1. $\tau_1 = \frac{1}{2} - \frac{1}{2\sqrt{3}}$, $\tau_2 = \frac{1}{2} + \frac{1}{2\sqrt{3}}$, $a_1 = a_2 = \frac{1}{2}$, $t_{ni} = t_{n-1}+\tau_i h_n$, i=1,2.

(6a) $$\begin{bmatrix} u_{n1} \\ u_{n2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} U_{n-1} + h_n \begin{bmatrix} \frac{1}{4} & \frac{1}{4}-\frac{1}{2\sqrt{3}} \\ \frac{1}{4}+\frac{1}{2\sqrt{3}} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} f(u_{n1},t_{n1}) \\ f(u_{n2},t_{n2}) \end{bmatrix},$$

(6b) $$U_n = U_{n-1} + \frac{h_n}{2}[f(u_{n1},t_{n1})+f(u_{n2},t_{n2})].$$

This is the Hammer and Hollingsworth process of order 4 [21].

<u>Case K=2</u>. $\tau_1 = \frac{1}{2} - \frac{\sqrt{3}}{2\sqrt{5}}$, $\tau_2 = \frac{1}{2}$, $\tau_3 = \frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{5}}$, $a_1 = \frac{5}{18}$, $a_2 = \frac{8}{18}$, $a_3 = \frac{5}{18}$

$t_{ni} = t_{n-1} + \tau_i h_n$, i=1,2,3.

$$\text{(7a)}\quad \begin{bmatrix} u_{n1} \\ u_{n2} \\ u_{n3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U_{n-1} + h_n \begin{bmatrix} \frac{5}{36} & \frac{2}{9} - \frac{1}{\sqrt{15}} & \frac{5}{36} - \frac{1}{2\sqrt{15}} \\ \frac{5}{36} + \frac{\sqrt{5}}{8\sqrt{3}} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{5}}{8\sqrt{3}} \\ \frac{5}{36} + \frac{1}{2\sqrt{15}} & \frac{2}{9} + \frac{1}{\sqrt{15}} & \frac{5}{36} \end{bmatrix} \begin{bmatrix} f(u_{n1},t_{n1}) \\ f(u_{n2},t_{n2}) \\ f(u_{n3},t_{n3}) \end{bmatrix},$$

$$\text{(7b)}\quad U_n = U_{n-1} + \frac{h_n}{18}[5f(u_{n1},t_{n1}) + 8f(u_{n2},t_{n2}) + 5f(u_{n3},t_{n3})].$$

APPENDIX 5. MULTISTEP METHODS (L=K+1).

This family of examples corresponds to the numerical scheme (3.29) of Corollary 3.8 when L = K+1. The L^2 and nodal-convergence rates are both proportional to h^{K+1}. For simplicity we shall assume that $h_n = h$ = a constant.

Here the K+1 points $\tau_1 < \tau_2 < \dots < \tau_{K+1} \le 1$ are fixed a priori. The quadrature formula uses these points. So it is an interpolatory quadrature formula which integrates exactly polynomials in $P^K(0,1)$. Its weights are given by

$$\text{(1)}\quad a_i = \int_0^1 \psi_i(\tau)d\tau,\ i=1,\dots,K+1,$$

where $\psi_1,\psi_2,\dots,\psi_{K+1}$ in $P^K(0,1)$ are the Lagrange interpolating polynomials for the points $\{\tau_i,\ 1 \le i \le K+1\}$:

$$\text{(2)}\quad \psi_i(\tau) = \prod_{\substack{j=1\\ j\ne i}}^{K+1} \frac{\tau-\tau_j}{\tau_i-\tau_j},\quad 1 \le i \le K+1.$$

The general formula is

$$\text{(3a)}\quad u_{nk} = U_{n_k},\quad 1 \le k \le K+1,$$

$$\text{(3b)}\quad U_n = U_{n-1} + h \sum_{\ell=1}^{K+1} a_\ell f(U_{n_\ell}, t_{n_\ell}).$$

When there exist k such that $u_{nk} = U_n$, equation (3b) is implicit; otherwise (3b) is an explicit equation.

<u>Case K=0</u>. a) $\tau_1=0$, $n_1=n-1$, $a_1=1$.

$$\text{(4)}\quad \begin{cases} u_{n1} = U_{n-1},\quad t_{n1} = t_{n-1}, \\ U_n = U_{n-1} + h\, f(u_{n1},t_{n1})\ \ (=U_{n-1} + hf(U_{n-1},t_{n-1})). \end{cases}$$

This is the <u>Euler explicit</u> process of order 1.

b) $\tau_1 = 1$, $n_1 = n$, $a_1 = 1$.

$$\text{(5)}\quad \begin{cases} u_{n1} = U_n,\quad t_{n1} = t_n, \\ U_n = U_{n-1} + h\, f(u_{n1},t_{n1})\ \ (=U_{n-1} + hf(U_n,t_n)). \end{cases}$$

This is the <u>Euler implicit</u> process of order 1.

c) $\tau_1 = -\alpha$, $\alpha \ge 0$ an integer, $n_1 = n-1-\alpha$, $a_1 = 1$.

$$\text{(6)}\quad \begin{cases} u_{n1} = U_{n-1-\alpha},\quad t_{n1} = t_{n-1-\alpha}, \\ U_n = U_{n-1} + hf(u_{n1},t_{n1})\ \ (=U_{n-1} + hf(U_{n-\alpha-1},t_{n-\alpha-1})). \end{cases}$$

<u>Case K=1</u>. $\tau_1 < \tau_2 \le 1$, integers.

$$\text{(7)}\quad a_1 = \frac{\tau_2 - 1/2}{\tau_2-\tau_1},\quad a_2 = \frac{1/2-\tau_1}{\tau_2-\tau_1},$$

$$\text{(8)}\quad \begin{cases} u_{ni} = U_{n-1+\tau_i},\ i=1,2, \\ U_n = U_{n-1} + \frac{h}{\tau_2-\tau_1}\,[(\tau_2 - \frac{1}{2})f(U_{n-1+\tau_1},t_{n-1+\tau_1}) + (\frac{1}{2}-\tau_1)f(U_{n-1+\tau_2},t_{n-1+\tau_2})]. \end{cases}$$

For $\tau_1=0$ and $\tau_2=1$ we obtain one of the B.L.HULME [22,23]'s continuous processes:

$$\text{(9)}\quad \begin{cases} u_{n1} = U_{n-1},\quad u_{n2} = U_n, \\ U_n = U_{n-1} + \frac{h}{2}[f(U_{n-1},t_{n-1}) + f(U_n,t_n)]. \end{cases}$$

Case K=2. $\tau_1 < \tau_2 < \tau_3 \le 1$, integers.

$$(10)\begin{cases} a_1 = \dfrac{2-3(\tau_2+\tau_3)+6\tau_2\tau_3}{6(\tau_1-\tau_2)(\tau_1-\tau_3)} \\ a_2 = \dfrac{2-3(\tau_3+\tau_1)+6\tau_3\tau_1}{6(\tau_2-\tau_3)(\tau_2-\tau_1)} \\ a_3 = \dfrac{2-3(\tau_1+\tau_2)+6\tau_1\tau_2}{6(\tau_3-\tau_1)(\tau_3-\tau_2)} \end{cases}$$

$$(11)\begin{cases} u_{ni} = U_{n-1+\tau_i}, \quad i=1,2,3, \\ U_n = U_{n-1} + h \sum_{i=1}^{3} a_i f(U_{n-1+\tau_i}, t_{n-1+\tau_i}). \end{cases}$$

When $\tau_1=-1$, $\tau_2=0$, $\tau_3=1$ we obtain

$$(12)\quad U_n = U_{n-1} + \frac{h}{12}[-f(U_{n-2},t_{n-2})+8f(U_{n-1},t_{n-1})+5f(U_n,t_n)].$$

This is the Adams-Moulton scheme of order 3. When $\tau_1 = -2$, $\tau_2 = -1$, $\tau_3 = 0$, we obtain

$$(13)\quad U_n = U_{n-1} + \frac{h}{12}[5f(U_{n-3},t_{n-3})-16f(U_{n-2},t_{n-2})+23f(U_{n-1},t_{n-1})].$$

This is the Adams-Bashforth scheme of order 3.

Case K=3. $\tau_1 < \tau_2 < \tau_3 < \tau_4 \le 1$, integers

$$(14)\begin{cases} a_1 = \dfrac{3-4(\tau_2+\tau_3+\tau_4)+6(\tau_2\tau_3+\tau_3\tau_4+\tau_4\tau_2)-12\tau_2\tau_3\tau_4}{12(\tau_1-\tau_2)(\tau_1-\tau_3)(\tau_1-\tau_4)} \\ a_2 = \dfrac{3-4(\tau_3+\tau_4+\tau_1)=6(\tau_3\tau_4+\tau_4\tau_1+\tau_1\tau_3)-12\tau_3\tau_4\tau_1}{12(\tau_2-\tau_3)(\tau_2-\tau_4)(\tau_2-\tau_1)} \\ a_3 = \dfrac{3-4(\tau_4+\tau_1+\tau_2)+6(\tau_4\tau_1+\tau_1\tau_2+\tau_2\tau_4)-12\tau_4\tau_1\tau_2}{12(\tau_3-\tau_4)(\tau_3-\tau_1)(\tau_3-\tau_2)} \\ a_4 = \dfrac{3-4(\tau_1+\tau_2+\tau_3)+6(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1)-12\tau_1\tau_2\tau_3}{12(\tau_4-\tau_1)(\tau_4-\tau_2)(\tau_4-\tau_3)} \end{cases}$$

$$(15)\begin{cases} u_{ni} = U_{n-1+\tau_i}, \quad i=1,2,3,4, \\ U_n = U_{n-1} + h \sum_{i=1}^{4} a_i f(U_{n-1+\tau_i}, t_{n-1+\tau_i}). \end{cases}$$

The parameters $\tau_1 = -2$, $\tau_2 = -1$, $\tau_3 = 0$, $\tau_4 = 1$ yield the Adams-Moulton process of order 4,

$$(16)\quad U_n = U_{n-1} + \frac{h}{24}[f(U_{n-3},t_{n-3})-5f(U_{n-2},t_{n-2})+19f(U_{n-1},t_{n-1})+9f(U_n,t_n)].$$

Similarly, the parameters $\tau_1=-3$, $\tau_2=-2$, $\tau_3=-1$, $\tau_4=0$ yield the Adams-Bashford scheme of order 4,

$$(17)\quad U_n = U_{n-1} + \frac{h}{24}[-9f(U_{n-4},t_{n-4})+37f(U_{n-3},t_{n-3})-59f(U_{n-2},t_{n-2})+55f(U_{n-1},t_{n-1})].$$

APPENDIX 6. CONTINUOUS METHODS (L=2, K≥1).

This family of methods corresponds to the numerical scheme (3.29) of Corollary 3.8 when L=2 and $\tau_1=0$, $\tau_2=1$ (that is, the polynomial approximation is now continuous at each node). This is B.L.HULME [22,23]'s process. The L^2 and nodal-convergence rates are respectively proportional to h^{K+1} and h^{2K}.

The K+1 points

$$(1)\quad 0 = \tau_1 < \tau_3 < \tau_4 < \dots < \tau_{K+1} < \tau_2 = 1$$

are the abscissae in [0,1] of the (K+1)-point Gauss-Lobatto quadrature formula with weights $a_1, a_3, a_4, \dots, a_{K+1}, a_2$.

The general formula is

(2a) $u_{n1} = U_{n-1}$, $t_{n1} = t_{n-1}$,

(2b) $u_{n2} = U_n$, $t_{n2} = t_n$,

$$(2c)\begin{cases} u_{nk} = U_{n-1}+(U_n-U_{n-1})\beta_2(\tau_k) + h_n \frac{a_1}{a_k} f(U_{n-1},t_{n-1})\psi_k(0) \\ \qquad + h_n \sum_{\ell=3}^{K+1} \frac{a_\ell \psi_k(\tau_\ell)}{a_k} f(u_{n\ell},t_{n\ell}), \qquad 3 \le k \le K+1, \\ U_n = U_{n-1} + h_n\{a_1 f(U_{n-1},t_{n-1})+a_2 f(U_n,t_n)+ \sum_{\ell=3}^{K+1} a_\ell f(u_{n\ell},t_{n\ell})\ , \end{cases}$$

where

$$(3) \quad \beta_2(\tau_k) = -\frac{a_2}{a_k}\theta_k(1), \qquad 3 \le k \le K+1.$$

For K=1 formula (2) reduces to

$$(4)\begin{cases} u_{n1} = U_{n-1}, \quad t_{n1} = t_{n-1}, \\ u_{n2} = U_n, \quad t_{n2} = t_n, \\ U_n = U_{n-1} + \frac{h_n}{2}[f(U_{n-1},t_{n-1})+f(U_n,t_n)]. \end{cases}$$

For $K \ge 2$ formula (2) can be rewritten as follows:

$$(5a)\begin{cases} u_{n1} = U_{n-1}, \quad t_{n1} = t_{n-1}, \\ u_{n2} = U_n, \quad t_{n2} = t_n \end{cases}$$

$$(5b)\begin{cases} u_{nk} = U_{n-1} + h_n \frac{a_1}{a_k}[\psi_k(0)-a_2\theta_k(1)]f(U_{n-1},t_{n-1}) \\ \qquad + h_n \sum_{\ell=3}^{K+1} \frac{a_\ell}{a_k}[\psi_k(\tau_\ell)-a_2\theta_k(1)]f(u_{n\ell},t_{n\ell}) + h_n \frac{a_2}{a_k}[-a_2\theta_k(1)]f(U_n,t_n), \\ \qquad 3 \le k \le K+1, \\ U_n = U_{n-1} + h_n\{a_1 f(U_{n-1},t_{n-1})+a_2 f(U_n,t_n) + \sum_{\ell=3}^{K+1} a_\ell f(u_{n\ell},t_{n\ell})\ . \end{cases}$$

Note that formulae (4) and (5) are different from the formula given by B.L.HULME [22],[23] since he used a K-point Gauss-Legendre formula instead of a (K+1)-point Gauss-Radau formula. However, in both cases we solve K equations in K unknowns.

For $K=2$, $\tau_1=0$, $\tau_2=1$, $\tau_3=1/2$, $a_1 = a_2 = 1/6$ and $a_3 = 4/6$. So formula (5) reduces to

$$(6a) \quad u_{n1} = U_{n-1}, \quad u_{n2} = U_n,$$

$$(6b)\begin{cases} u_{n3} = U_{n-1} + \frac{h_n}{24}[5f(U_{n-1},t_{n-1})+8f(u_{n3},t_{n3})-f(U_n,t_n)], \\ U_n = U_{n-1} + \frac{h_n}{6}[f(U_{n-1},t_{n-1})+4f(u_{n3},t_{n3})+f(U_n,t_n)]. \end{cases}$$

APPENDIX 7. HYBRID METHODS (L=1, τ_1=1 or 0).

Consider the family of examples corresponding to the numerical scheme (3.29) of Corollary 3.8 with L=1 and τ_1=1 or 0. The L^2 and nodal-convergence rates are respectively proportional to h^{K+1} and h^{2K+1}.

Case $\tau_1=1$ ($n_1=n$)

The τ_i's are the abscissae of the (K+1)-point Gauss-Radau quadrature formula:

$$(1) \quad 0 < \tau_2 < \tau_3 < \ldots < \tau_{K+1} < \tau_1 = 1.$$

This formula is exact for polynomials in $P^{2K}(0,1)$. The polynomials $\theta_2,\theta_3,\ldots,\theta_{K+1}$ in $P^{K-1}(0,1)$ are the Lagrange interpolating polynomials associated with the points $\tau_2,\tau_3,\ldots,\tau_{K+1}$:

$$(2) \quad \theta_i(\tau) = \prod_{\substack{j=2 \\ j\ne i}}^{K+1} \frac{\tau-\tau_j}{\tau_i-\tau_j}, \quad i=2,3,\ldots,K+1,$$

and

$$(3)\begin{cases} \psi_i(\tau) = \int_\tau^1 \theta_i(\sigma)d\sigma \\ t_{ni} = t_{n-1}+\tau_i h_n \end{cases} \quad i=2,3,\ldots,K+1;$$

This yields the following numerical scheme

$$(4a) \quad u_{n1} = U_n, \quad t_{n_1} = t_n,$$

$$(4b)\quad \begin{cases} u_{nk} = U_{n-1} + \beta_1(\tau_k)(U_n - U_{n-1}) + h_n \sum_{\ell=2}^{K+1} \frac{a_\ell \psi_k(\tau_\ell)}{a_k} f(u_{n\ell}, t_{n\ell}), \\ U_n = U_{n-1} + h_n[a_1 f(U_n, t_n) + \sum_{\ell=2}^{K+1} a_\ell f(u_{n\ell}, t_{n\ell})]. \end{cases}$$

Recall that $\beta_1 \in P^K(0,1) \cap [P^{K-1}(0,1)]^\perp$ and that $\beta_1(\tau_1) = 1$. Hence $\beta_1 = \pi$, the Legendre polynomial of degree K on [0,1] normalized to 1 in 1. So equation (4) is equivalent to

$$(5a)\quad \begin{bmatrix} u_{n2} \\ \vdots \\ u_{nk} \\ \vdots \\ u_{n,K+1} \\ \hline U_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \hline 1 \end{bmatrix} U_{n-1} + h_n \left[\begin{array}{c|c} a_\ell\left(\frac{\psi_k(\tau_\ell)}{a_k} + \pi(\tau_k)\right) & a_1 \pi(\tau_k) \\ \hline a_\ell & a_1 \end{array}\right] \begin{bmatrix} f(u_{n2}, t_{n2}) \\ \vdots \\ f(u_{n\ell}, t_{n\ell}) \\ \vdots \\ f(u_{n,K+1}, t_{n,K+1}) \\ \hline f(U_n, t_n) \end{bmatrix},$$

$$(5b)\quad u_{n1} = U_n.$$

The above family of numerical schemes is the one of LESAINT and RAVIART [25] with a (K+1)-point Gauss-Radau quadrature formula. It also corresponds to the case $\alpha_j = 1$, $j=0,1,2,\ldots$ in DELFOUR, HAGER and TROCHU [14] where examples can be found for K=0,1,2,3.

Case $\tau_1 = 0$ $(n_1 = n-1)$

Here the τ_i's are the abscissae of the (K+1)-point Gauss-Radau quadrature formula:

$$(6)\quad 0 = \tau_1 < \tau_2 < \ldots < \tau_{K+1} < 1.$$

This formula integrates exactly polynomials in $P^{2K}(0,1)$. The polynomials $\theta_2, \ldots, \theta_{K+1}$, $\psi_2, \ldots, \psi_{K+1}$ and the points $t_{n2}, \ldots, t_{n,K+1}$ are defined in an analogous fashion to (2) and (3). The resulting scheme is the following:

$$(7a)\quad u_{n1} = U_{n-1}, \quad t_{n1} = t_{n-1},$$

$$(7b)\quad u_{nk} = U_{n-1} + h_n\left\{\frac{a_1}{a_k}\psi_k(0) f(U_{n-1}, t_{n-1}) + \sum_{\ell=2}^{K+1} \frac{a_\ell}{a_k}\psi_k(\tau_\ell) f(u_{n\ell}, t_{n\ell})\right\}, \quad 2 \le k \le K+1,$$

$$(7c)\quad U_n = U_{n-1} + h_n\left\{a_1 f(U_{n-1}, t_{n-1}) + \sum_{\ell=2}^{K+1} a_\ell f(u_{n\ell}, t_{n\ell})\right\}.$$

This family of numerical schemes corresponds to the choice $\alpha_j = 0$, $j=0,1,2,\ldots$, in DELFOUR, HAGER and TROCHU [14] with a (K+1)-point Gauss-Radau quadrature formula $(\tau_1 = 0)$. Examples for K=0,1,2,3 can be found therein.